Nonlinear Dynamics

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Reference: G. Aruldhas

11.5 PHASE PLANE ANALYSIS OF DYNAMICAL SYSTEMS

For understanding the dynamics of linear and nonlinear systems, the description of its behaviour in phase space is quite useful. The two independent variables (x, p_x) here $(x, y = \dot{x})$, define the space in which the solution moves. For a particle having only one independent variable, the phase space is only two-dimensional and hence it is often referred to as the **phase plane**. At any time the value of the phase space co-ordinates (x, y) completely defines the state of the system. For a system having n independent variables $x_1, x_2, \ldots, x_n$, each variable can be thought of as an independent phase space co-ordinate in the associated n-dimensional phase space. A given solution to the equations of motion will map out a smooth curve in the phase plane as a function of time. This is called a **phase curve** or **phase trajectory** and the motion along it is called the **phase flow**. Because of the unique properties of solutions to differential equations, different phase space trajectories do not cross each other. A picture made up of sets of phase curves is often called a **phase portrait**.

Phase Curve of Simple Harmonic Oscillator

To illustrate the various concepts, we make use of the familiar harmonic oscillator problem as given in Eq. (11.2). In its first integral, Eq. (11.6), I_1 is simply total energy. From Eq. (11.6)

$$\frac{y^2}{2E} + \frac{\omega_0^2}{2E} x^2 = 1 \tag{11.42}$$

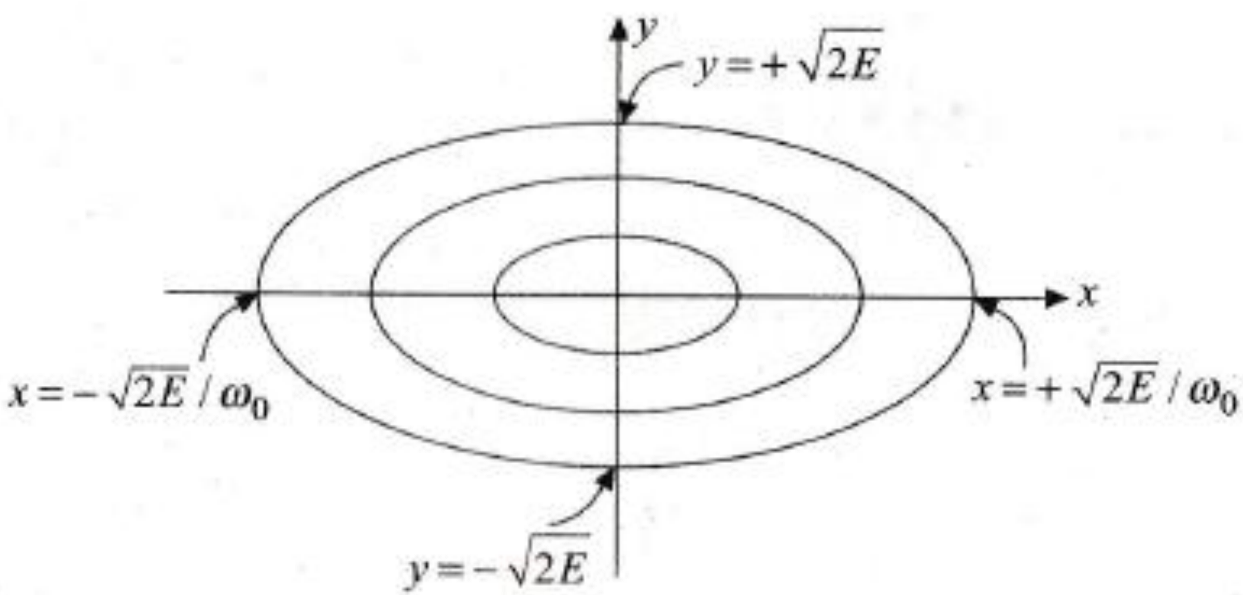

Fig. 11.1 Phase curves for the simple harmonic oscillator.

Clearly the phase trajectories are concentric ellipses (See Fig. 11.1). The semi-major and minor axes can easily be determined as detailed below. At the turning points of the ellipse, $y = \dot{x} = 0$. Hence, from Eq. (11.42)

$$\omega_0^2 x^2 = 2E \qquad \text{or} \qquad x = \pm\sqrt{2E}/\omega_0$$

When $x = 0$, again from Eq. (10.42)

$$y^2 = 2E \qquad \text{or} \qquad y = \pm\sqrt{2E}$$

Hence, the semi-major axis a and semi-minor axis b are given by

The origin of the phase plane $x = y = 0$ corresponds to an obvious equilibrium point of the motion. Thus, the existence of a constant first integral has provided a definite geometrical constraint on the phase flow.

Phase Curve of Damped Oscillator

Next we shall consider oscillations with damping forces proportional to velocity. The equation of the oscillator is then

$$\ddot{x} + 2b\,\dot{x} + \omega_0^2 x = 0 \tag{11.44}$$

This is equivalent to the coupled equations

$$y = \dot{x} \qquad \text{and} \qquad \dot{y} = -\,\omega_0^2 x - 2by \tag{11.45}$$

The trial solution

$$x = e^{\lambda t} \tag{11.45a}$$

in Eq. (11.44) gives the equation

$$e^{\lambda t}\left(\lambda^2 + 2b\lambda + \omega_0^2\right) = 0 \quad \text{or} \quad \lambda^2 + 2b\lambda + \omega_0^2 = 0$$

The quadratic in λ gives the roots

$$\lambda_1, \lambda_2 = -\,b \pm \sqrt{b^2 - \omega_0^2} \tag{11.46}$$

When $b^2 < \omega_0^2$, the roots λ_1 and λ_2 are complex and the motion is oscillation with decreasing amplitude. The solution just spirals into the equilibrium point at the origin at a rate depending on the damping coefficient b (See Fig. 11.2a).

In the case of $b^2 > \omega_0^2$ or $b^2 = \omega_0^2$, the solution is aperiodic damped motion. Phase trajectories for these cases are obtained numerically and are given in Figs. 11.2 (b) and (c). The trajectories in these two cases approach the origin.

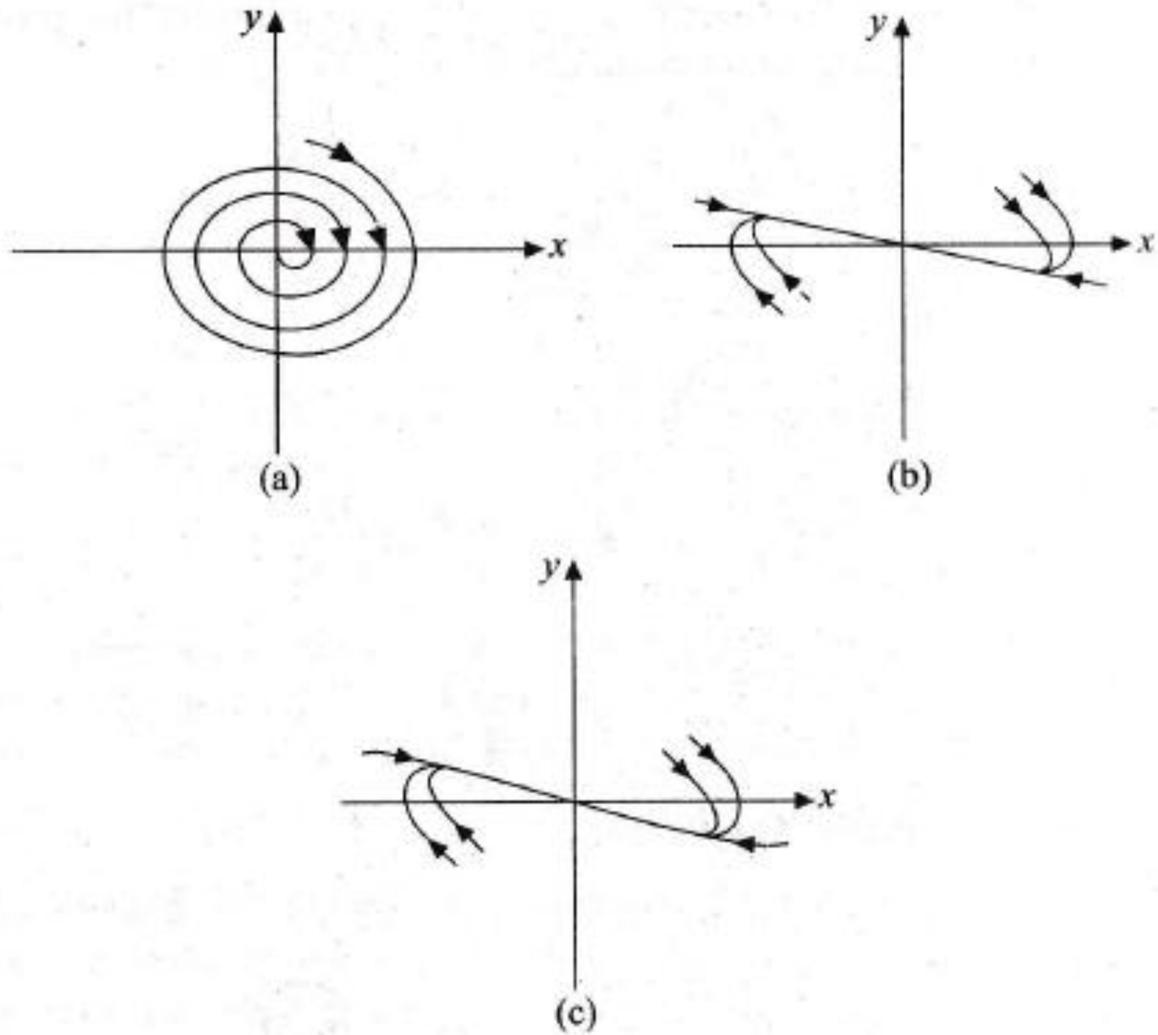

Fig. 11.2 Phase curves of a damped oscillator: (a) $b^2 < \omega_0^2$. The solution spirals into the equilibrium point; (b) $b^2 > \omega_0^2$; (c) $b^2 = \omega_0^2$. In (b) and (c) the trajectories approach the origin.

11.6 PHASE PORTRAIT OF THE PENDULUM

One of the most studied examples is the simple pendulum. Its time evolution equation for the displacement is given by Eq. (11.26). To have uniformity in notation with the previous section, we denote the variables θ and $\dot{\theta}$ as x and y, respectively. In the new notation Eq. (11.28) takes the form

$$\dot{x} = y \qquad \text{and} \qquad \dot{y} = -\,\omega_0^2 \sin x \tag{11.47}$$

Proceeding as in Section 11.4, in place of Eq. (11.30), we get

$$\frac{1}{2} y^2 = \omega_0^2 \cos x + E' \tag{11.48}$$

where E' is the scaled total energy. The phase space diagram of the pendulum is shown in Fig. 11.3.

For very small energies the pendulum will just oscillate about the equilibrium point $x = y = 0$ in nearly linear fashion. For small energies the phase space trajectories are ellipses centred on the origin. As the energy increases, the pendulum

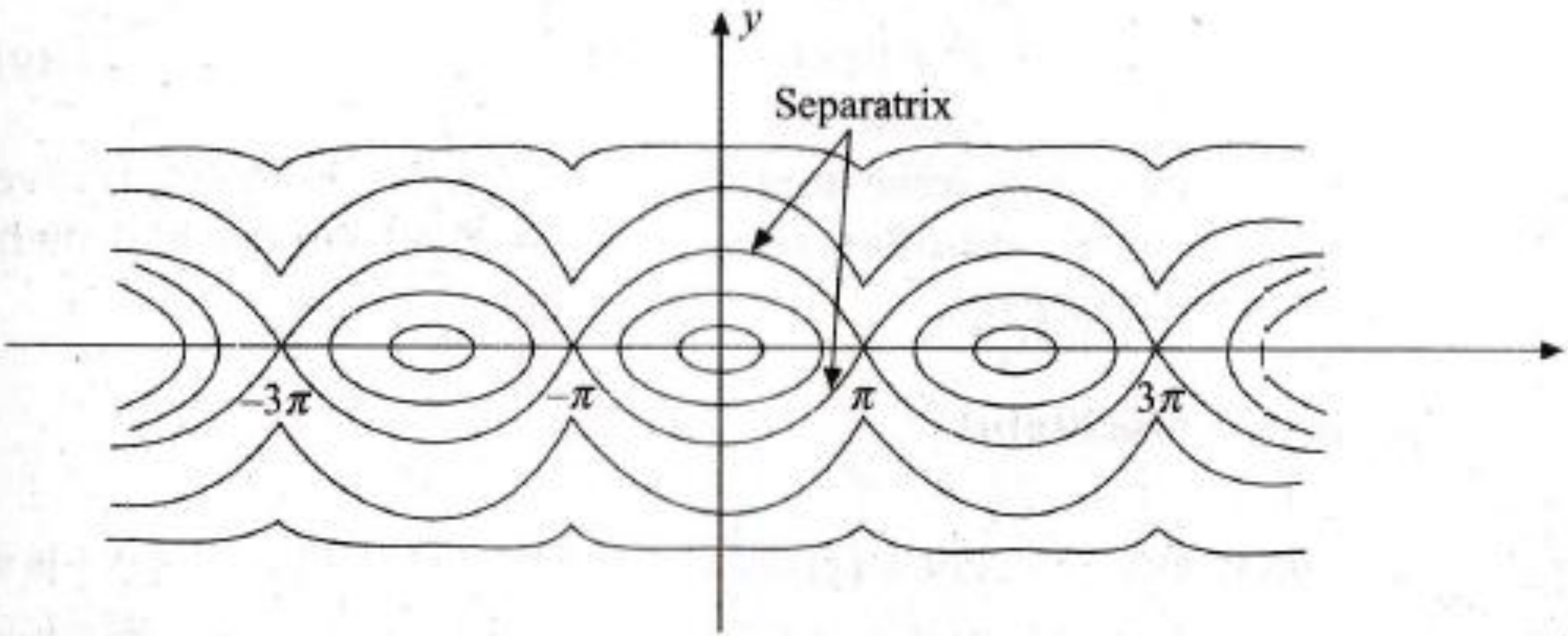

Fig. 11.3 Phase curves for the pendulum.

executes larger librations until finally a point is reached with the pendulum standing straight up with the mass directly above the point of pivot and starts to execute rotational motion. That means the pendulum has sufficient energy to swing from $x = 0$ to $x = \pm\pi$, the value of y for these two values of x being zero. When $y = 0$ and $x = \pm\pi$, from Eq. (11.48) $E' = \omega_0^2 = g/l$. In other words, the pendulum will just complete the circle if it has the energy $E' = g/l$. As the energy increases further and further, the rotational motion gets faster and faster. The point $x = \pm\pi$ with $y = 0$ is an equilibrium point, but an unstable one. This phase pattern will be repeated at every multiple of 2π to the left and right since the restoring force is periodic. Thus, at every $x = \pm 2n\pi$ there is a **stable equilibrium** point and at every $x = \pm(2n+1)\pi$ there is an **unstable equilibrium** point. These points marks a transition from librational motion to rotational motion and the phase curves change from closed to open ones. The open one corresponds to unbounded rotational motion.

The pair of space curves that separate the librational and rotational motions and that meet at the unstable equilibrium points is termed the **separatrix**. Inside the separatrices, the motion is completely periodic and oscillatory. Trajectories outside the separatrices correspond to **running modes** in which the pendulum has sufficient energy to swing over the top. One type of running mode has angular velocity $\dot{\theta} > 0$ (anticlockwise motion) and the other type has angular velocity $\dot{\theta} < 0$ (clockwise motion).